MAT 1847 - Lecture 17

Entropy of real quadratic polynomials

Milnor-Thurston, On iterated maps of the interval

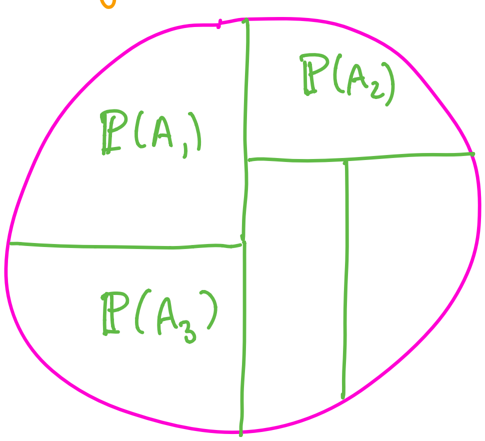
kneading theory

Preston, What you need to know to knead

Douady, Topological entropy of unimodal maps

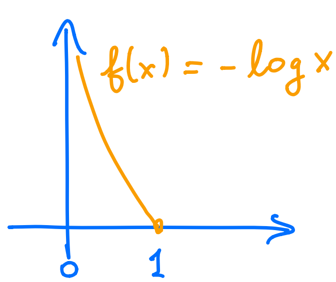
Entropy → measure-theoretic
→ topological

"average rate of information"



After doing one experiment, I get that my result follows into A_i

The "information" I gain is



$$I(A_i) := -\log \mathbb{P}(A_i)$$
$$I(x) := -\log \mathbb{P}(A_i(x))$$

If $\mathcal{P} = \{A_1, \dots, A_k\}$ is a partition of your state space X, then its entropy is

$$H(\mathcal{P}) := -\sum_{i=1}^{k} \mu(A_i) \log(A_i)$$
$$= \mathbb{E}_{\mu}[I(x)]$$

If μ is uniform on $\mathcal{P}$, $\mu(A_i) = \frac{1}{k}$

then $H(\mathcal{P}) := -\sum_{i=1}^{k} \frac{1}{k} \log\left(\frac{1}{k}\right) = \log k$

$$= \log \# \mathcal{P} .$$

Lemma

$$H_{\mu}(\mathcal{P}) \leq \log \# \mathcal{P}$$

Pf

$$f(t) = t \log t$$
$$f'(t) = \log t + 1$$
$$f''(t) = \frac{1}{t} > 0$$

Shannon '48: The entropy of the English language

DISCOVE...

$$f\left(\frac{x_1+\dots+x_k}{k}\right) \le \frac{\sum_{i=1}^{k} f(x_i)}{k}$$

$x_i = \mu(A_i)$

$$f\left(\frac{1}{k}\right) \le \cancel{\frac{1}{k}} \sum \mu(A_i) \log \mu(A_i)$$

$$\cancel{\frac{1}{k}} \log\left(\frac{1}{k}\right)$$

$$-\sum_i \mu(A_i) \log \mu(A_i) \le \log k \quad \square$$

$T : X \to X$ Dynamical System

$$\bigvee_{k=0}^{n-1} T^{-k}(\mathcal{P})$$

join

$$T^{-k}(\mathcal{P}) = \bigsqcup_{i=1}^{k} T^{-k}(A_i)$$

each element of $\bigvee_{k=0}^{n-1} T^{-k}(\mathcal{P})$

is $A_{i_1} \cap T^{-1}(A_{i_2}) \cap \dots \cap T^{-(n-1)}(A_{i_n})$

Fact if $\mathcal{A} \leq \mathcal{B}$ then $H(\mathcal{A}) \leq H(\mathcal{B})$.
finer

Def.: measure-theoretic entropy

$$h(\mathcal{P}, T) := \lim_{n \to \infty} \frac{1}{n} H\Big(\bigvee_{k=0}^{n-1} T^{-k}\mathcal{P}\Big) \quad (*)$$

$$h(T) := \sup_{\mathcal{P} \text{ partitions}} h(\mathcal{P}, T)$$

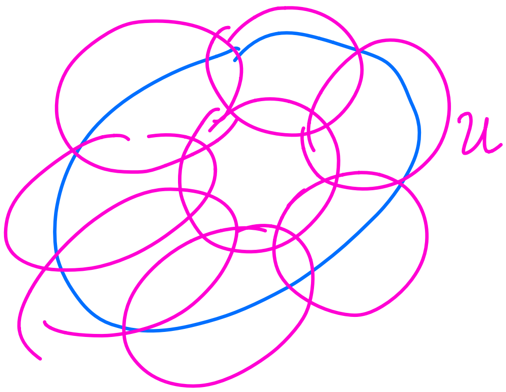
Fact the limit $(*)$ exists because
if $a_n := H\big(\bigvee_{k=0}^{n-1} T^{-k}\mathcal{P}\big)$ then

$$a_{n+m} \leq a_n + a_m \quad \text{so}$$

$$\lim_{n \to \infty} \frac{a_n}{n} = \inf_n \frac{a_n}{n} \quad \text{exists.}$$

Topological Entropy

Let $f : X \to X$ be a continuous map of a compact metric space.
Let $\mathcal{U} = (U_i)_{i=1}^{k}$ an open cover of X



$\#\mathcal{U} :=$ minimum cardinality of a subcover of $\mathcal{U}$ which still covers X

Def.: The TOPOLOGICAL ENTROPY of $f : X \circlearrowleft$ is:

$$h_{top}(\mathcal{U}, f) := \lim_{n} \frac{1}{n} \log \#\left(\bigvee_{k=0}^{n-1} f^{-k}\mathcal{U} \right)$$

$$h_{top}(f) := \sup_{\substack{\mathcal{U} \\ \text{cover of} \\ X}} h_{top}(\mathcal{U}, f)$$

Lemma If $X = X_1 \cup X_2$ with X_i cpt for each i, and $f(X_i) \subseteq X_i$ for $i = 1, 2$, then

$$h_{top}(f) = \max\left\{ h_{top}(f|_{X_1}), h_{top}(f|_{X_2}) \right\}$$

Lemma If $Y \subset X$ is closed and $f(Y) \subseteq Y$, and for any $x \in X$ the distance

$$d(f^n(x), Y) \to 0 \quad \text{uniformly on } X \setminus Y,$$

then $h(f|_Y) = h(f|_X)$

E.g.: $f(x) = \frac{x}{2}$, $X = [-1, 1]$

$$h_{top}(f) = h_{top}(f|_{\{0\}}) = 0$$

If $\mathcal{U}$ is attracting basin, $h_{top}(f|_{\mathcal{U}}) = 0$

If f is a hyperbolic rational map, then all components of its Fatou set are attracting basins, hence

$$h_{top}(f) = \max\{h(f, J(f)), h(f, \Omega(\alpha_i))\}$$

$$\left[\hat{\mathbb{C}} = J(f) \cup \bigcup_i \Omega(\alpha_i)\right]$$

α_i = attracting periodic cycles

But $h_{top}(f, \Omega(\alpha_i)) = h_{top}(f, \{\alpha_i\}) = 0$

So

$$h_{top}(f, \hat{\mathbb{C}}) = h_{top}(f, J(f)).$$

Lemma Let X, Y be compact spaces, $f : X \circlearrowleft$, $g : Y \circlearrowleft$ and $\pi : Y \to X$ is a continuous, surjective map with $f \circ \pi = \pi \circ g$.

Then $h_{top}(f) \leq h_{top}(g)$

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "\pi"'] & Y \arrow[d, "\pi"] \\ X \arrow[r, "f"] & X \end{tikzcd}$$

and if there exists m s.t.

$\# \pi^{-1}(x) \leq m$ for all $x \in X$, then

$$h_{top}(f) = h_{top}(g).$$

Proof ① $h_{top}(g) \leq h_{top}(f) + \log m$

because for each cover $\mathcal{U}$ of X you have

$$\# \pi^{-1}\left(\bigvee_{k=0}^{n-1} \mathcal{U}\right) \leq m^n \, \#\left(\bigvee_{k=0}^{n-1} \mathcal{U}\right)$$

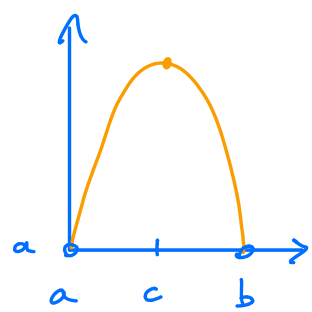
(2) $$h_{top}(f^k) = k\, h_{top}(f)$$

(3)
$$h_{top}(g^k) \leq h_{top}(f^k) + \log m$$

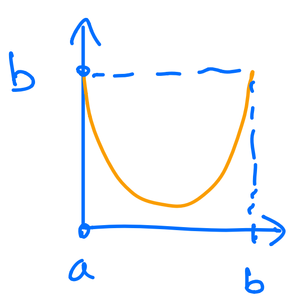
$$k\, h_{top}(g) \leq k\, h_{top}(f) + \log m$$

$$h_{top}(g) \leq h_{top}(f) + \underbrace{\frac{\log m}{k}}_{\downarrow 0} \text{ as } k \to \infty .$$

Unimodal Maps

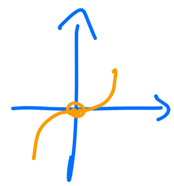


Def.: A <u>unimodal map</u> is a continuous interval map $f: I \to I = [a,b]$ with $f(a) = f(b) = a$ or b so that there is $c \in [a,b]$ so that f is monotone on $[a,c]$ and $[c,b]$ and monotonicity changes at c.



c is called a (TOPOLOGICAL) CRITICAL POINT or a TURNING POINT.

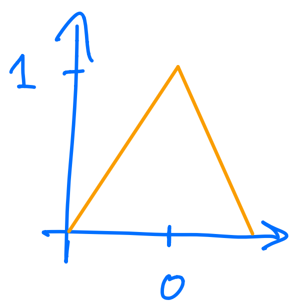
Note: for $f(x) = x^3$, $x = 0$ is not a turning point



E.g.: ① $f_c(x) = x^2 + c$, $c \in \mathbb{R}$

$f_a(x) = ax(1-x)$, $0 \leq a \leq 4$

② tent maps of slope λ



$$T_\lambda(x) = \begin{cases} -\lambda x + 1, & x \geq 0 \\ \lambda x + 1, & x < 0 \end{cases}$$

Theorem

If f is a tent map of slope $\pm\lambda$ then $h_{top}(f) = \log \lambda$.

Theorem (Milnor-Thurston)

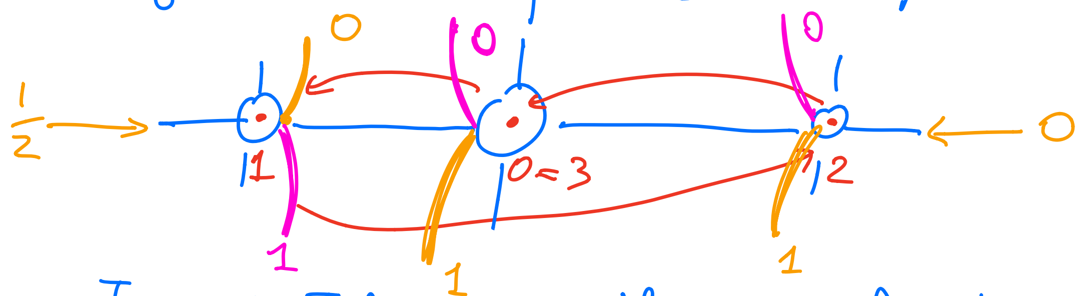
The topological entropy function

$$c \longmapsto h_{top}(f_c), \quad c \in [-2, \tfrac{1}{4}]$$

is continuous and monotone (decreasing).

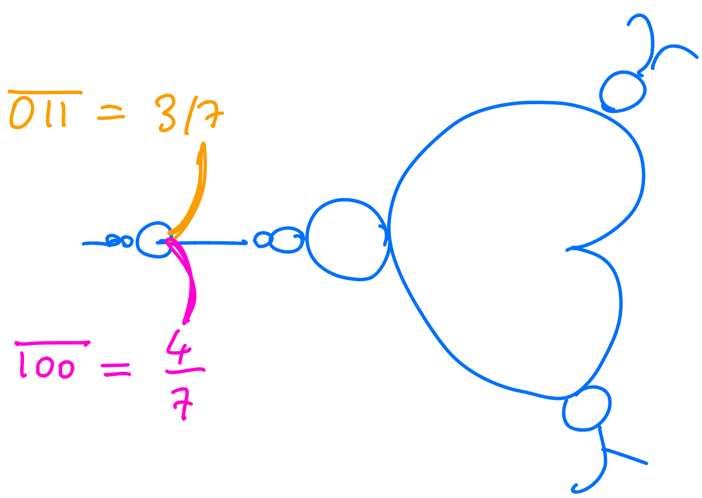
E.g.: $\frac{3}{7} = 0.\overline{011} = \frac{0\cdot 2^2 + 1\cdot 2^1 + 1\cdot 2^0}{2^3 - 1}$

is the ray landing at the root of the airplane component.



Take in Julia set the ray landing at the root of the Fatou cpt containing the critical value.

Record 0 if ray lies in top half plane and 1 if ray lies in bottom half plane.

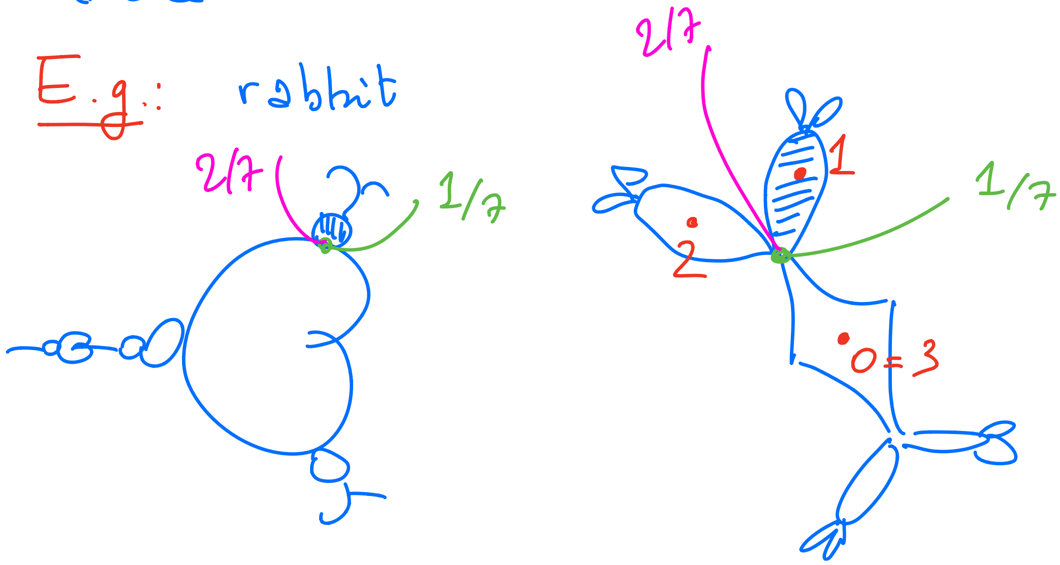


In the M-set, the angle $\frac{3}{7}$ (angle of ray landing at critical value in dynamical plane) lands at the root of the airplane component,

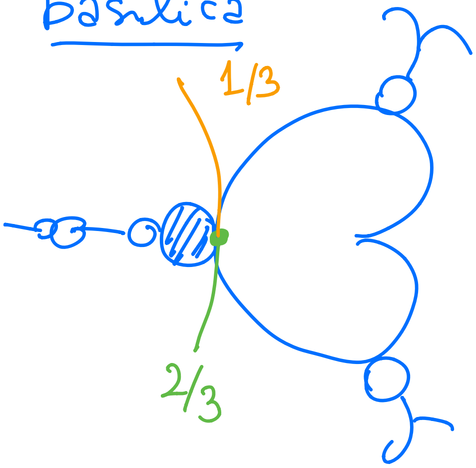
Theorem (Douady - Hubbard)

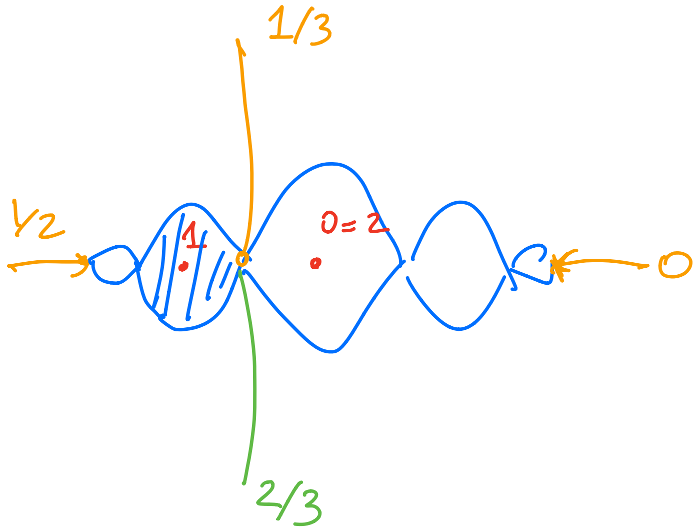
Given a hyperbolic component of the M-set, there are two rays of angles θ_1, θ_2 landing at it. The rays with these same angles land, in the dynamical plane of the polynomial which is at the center of the component, together on the boundary of the Fatou component containing the critical value.

E.g.: rabbit



basilica





$$4x \equiv x \pmod 1$$